AH National Qualifications 2016

X747/77/11 Mathematics

THURSDAY, 12 MAY

9:00 AM – 12:00 NOON

Total marks — 100

Attempt ALL questions.

You may use a calculator.

Full credit will be given only to solutions which contain appropriate working.

State the units for your answer where appropriate.

Answers obtained by readings from scale drawings will not receive any credit.

Write your answers clearly in the answer booklet provided. In the answer booklet, you must clearly identify the question number you are attempting.

Use **blue** or **black** ink.

Before leaving the examination room you must give your answer booklet to the Invigilator; if you do not you may lose all the marks for this paper.

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FORMULAE LIST

Standard derivatives

$f(x)$	$f'(x)$
$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}$
$\cos^{-1} x$	$-\frac{1}{\sqrt{1-x^2}}$
$\tan^{-1} x$	$\frac{1}{1+x^2}$
$\tan x$	$\sec^2 x$
$\cot x$	$-\operatorname{cosec}^2 x$
$\sec x$	$\sec x \tan x$
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$
$\ln x$	$\frac{1}{x}$
e^x	e^x

Standard integrals

$f(x)$	$\int f(x)\,dx$
$\sec^2(ax)$	$\frac{1}{a}\tan(ax)+c$
$\frac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1}\left(\frac{x}{a}\right)+c$
$\frac{1}{a^2+x^2}$	$\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+c$
$\frac{1}{x}$	$\ln\lvert x\rvert+c$
e^{ax}	$\frac{1}{a}e^{ax}+c$

Summations

(Arithmetic series) $$S_n = \frac{1}{2}n\left[2a+(n-1)d\right]$$

(Geometric series) $$S_n = \frac{a(1-r^n)}{1-r}$$

$$\sum_{r=1}^{n} r = \frac{n(n+1)}{2}, \quad \sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}, \quad \sum_{r=1}^{n} r^3 = \frac{n^2(n+1)^2}{4}$$

Binomial theorem

$$(a+b)^n = \sum_{r=0}^{n}\binom{n}{r}a^{n-r}b^r \quad \text{where} \quad \binom{n}{r} = {}^nC_r = \frac{n!}{r!(n-r)!}$$

Maclaurin expansion

$$f(x) = f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \frac{f^{iv}(0)x^4}{4!} + \ldots$$

FORMULAE LIST (continued)

De Moivre's theorem

$$\left[r(\cos\theta + i\sin\theta)\right]^n = r^n\left(\cos n\theta + i\sin n\theta\right)$$

Vector product

$$\mathbf{a}\times\mathbf{b} = |\mathbf{a}||\mathbf{b}|\sin\theta\,\hat{\mathbf{n}} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \mathbf{i}\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \mathbf{j}\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \mathbf{k}\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

Matrix transformation

Anti-clockwise rotation through an angle, θ, about the origin, $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

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MARKS

Total marks — 100

Attempt ALL questions

1. (a) Differentiate $y = x\tan^{-1}2x$. **3**

(b) Given $f(x) = \dfrac{1-x^2}{1+4x^2}$, find $f'(x)$, simplifying your answer. **3**

(c) A curve is given by the parametric equations

$$x = 6t \text{ and } y = 1 - \cos t.$$

Find $\dfrac{dy}{dx}$ in terms of t. **2**

2. A geometric sequence has second and fifth terms 108 and 4 respectively.

(a) Calculate the value of the common ratio. **3**

(b) State why the associated geometric series has a sum to infinity. **1**

(c) Find the value of this sum to infinity. **2**

3. Write down and simplify the general term in the binomial expansion of $\left(\dfrac{3}{x} - 2x\right)^{13}$.

Hence, or otherwise, find the term in x^9. **5**

4. Below is a system of equations:

$$x + 2y + 3z = 3$$
$$2x - y + 4z = 5$$
$$x - 3y + 2\lambda z = 2$$

Use Gaussian elimination to find the value of λ which leads to redundancy. **4**

5. Prove **by induction** that

$$\sum_{r=1}^{n} r(3r-1) = n^2(n+1), \quad \forall n \in \mathbb{N}.$$ **4**

MARKS

6. Find Maclaurin expansions for $\sin 3x$ and e^{4x} up to and including the term in x^3.

Hence obtain an expansion for $e^{4x}\sin 3x$ up to and including the term in x^3. **6**

7. A is the matrix $\begin{pmatrix} 2 & 0 \\ \lambda & -1 \end{pmatrix}$.

(a) Find the determinant of matrix A. **1**

(b) Show that A^2 can be expressed in the form $pA + qI$, stating the values of p and q. **3**

(c) Obtain a similar expression for A^4. **2**

8. Let $z = \sqrt{3} - i$.

(a) Plot z on an Argand diagram. **1**

(b) Let $w = az$ where $a > 0$, $a \in \mathbb{R}$.

Express w in polar form. **2**

(c) Express w^8 in the form $ka^n\left(x + i\sqrt{y}\right)$ where $k, x, y \in \mathbb{Z}$. **3**

9. Obtain $\int x^7 (\ln x)^2 \, dx$. **6**

10. For each of the following statements, decide whether it is true or false.

If true, give a proof; if false, give a counterexample.

A. If a positive integer p is prime, then so is $2p + 1$.

B. If a positive integer n has remainder 1 when divided by 3, then n^3 also has remainder 1 when divided by 3. **4**

11. The height of a cube is increasing at the rate of $5\,\text{cm s}^{-1}$.

Find the rate of increase of the volume when the height of the cube is 3 cm. **4**

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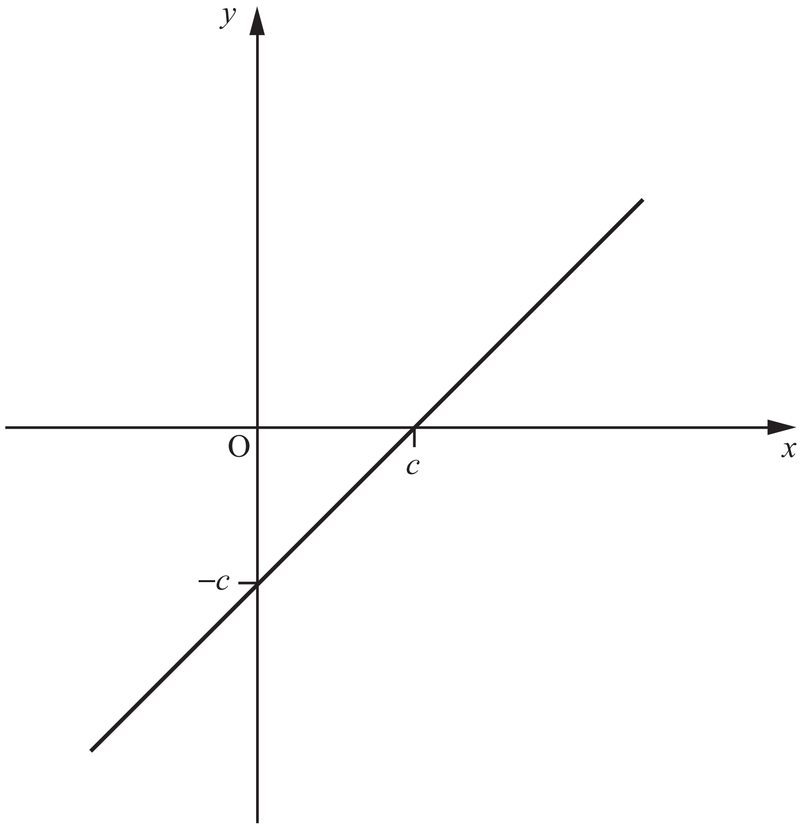
12. Below is a diagram showing the graph of a linear function, $y = f(x)$.



On separate diagrams show:

(a) $y = |f(x) - c|$ **2**

(b) $y = |2f(x)|$ **2**

13. Express $\frac{3x+32}{(x+4)(6-x)}$ in partial fractions and hence evaluate

$$\int_3^4 \frac{3x+32}{(x+4)(6-x)}\,dx.$$

Give your answer in the form $\ln\left(\frac{p}{q}\right)$. **9**

MARKS

14. Two lines L_1 and L_2 are given by the equations:

$$L_1: \quad x = 4 + 3\lambda, \quad y = 2 + 4\lambda, \quad z = -7\lambda$$

$$L_2: \quad \frac{x-3}{-2} = \frac{y-8}{1} = \frac{z+1}{3}$$

(a) Show that the lines L_1 and L_2 intersect and find the point of intersection. **5**

(b) Calculate the obtuse angle between the lines L_1 and L_2. **4**

15. Solve the differential equation

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 12x^2 + 2x - 5$$

given $y = -6$ and $\frac{dy}{dx} = 3$, when $x = 0$. **10**

16. A beaker of liquid was placed in a fridge.

The rate of cooling is given by

$$\frac{dT}{dt} = -k\left(T - T_F\right), \quad k > 0,$$

where T_F is the constant temperature in the fridge and T is the temperature of the liquid at time t.

- The constant temperature in the fridge is 4 °C.
- When first placed in the fridge, the temperature of the liquid was 25 °C.
- At 12 noon, the temperature of the liquid was 9·8 °C.
- At 12:15 pm, the temperature of the liquid had dropped to 6·5 °C.

At what time, to the nearest minute, was the liquid placed in the fridge? **9**

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