National Qualifications 2017

X747/77/11 Mathematics

FRIDAY, 5 MAY

9:00 AM – 12:00 NOON

Total marks — 100

Attempt ALL questions.

You may use a calculator.

Full credit will be given only to solutions which contain appropriate working.

State the units for your answer where appropriate.

Answers obtained by readings from scale drawings will not receive any credit.

Write your answers clearly in the answer booklet provided. In the answer booklet, you must clearly identify the question number you are attempting.

Use **blue** or **black** ink.

Before leaving the examination room you must give your answer booklet to the Invigilator; if you do not, you may lose all the marks for this paper.

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FORMULAE LIST

Standard derivatives	
$f(x)$	$f'(x)$
$\sin^{-1} x$	$\dfrac{1}{\sqrt{1-x^2}}$
$\cos^{-1} x$	$-\dfrac{1}{\sqrt{1-x^2}}$
$\tan^{-1} x$	$\dfrac{1}{1+x^2}$
$\tan x$	$\sec^2 x$
$\cot x$	$-\operatorname{cosec}^2 x$
$\sec x$	$\sec x \tan x$
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$
$\ln x$	$\dfrac{1}{x}$
e^x	e^x

Standard integrals	
$f(x)$	$\int f(x)\,dx$
$\sec^2(ax)$	$\dfrac{1}{a}\tan(ax)+c$
$\dfrac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1}\left(\dfrac{x}{a}\right)+c$
$\dfrac{1}{a^2+x^2}$	$\dfrac{1}{a}\tan^{-1}\left(\dfrac{x}{a}\right)+c$
$\dfrac{1}{x}$	$\ln\lvert x\rvert+c$
e^{ax}	$\dfrac{1}{a}e^{ax}+c$

Summations

(Arithmetic series) $$S_n = \frac{1}{2}n\left[2a+(n-1)d\right]$$

(Geometric series) $$S_n = \frac{a\left(1-r^n\right)}{1-r}$$

$$\sum_{r=1}^{n} r = \frac{n(n+1)}{2}, \quad \sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}, \quad \sum_{r=1}^{n} r^3 = \frac{n^2(n+1)^2}{4}$$

Binomial theorem

$$(a+b)^n = \sum_{r=0}^{n}\binom{n}{r}a^{n-r}b^r \quad \text{where} \quad \binom{n}{r} = {}^nC_r = \frac{n!}{r!(n-r)!}$$

Maclaurin expansion

$$f(x) = f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \frac{f^{iv}(0)x^4}{4!} + \ldots$$

De Moivre's theorem

$$\left[r(\cos\theta + i\sin\theta)\right]^n = r^n\left(\cos n\theta + i\sin n\theta\right)$$

Vector product

$$\mathbf{a}\times\mathbf{b} = |\mathbf{a}||\mathbf{b}|\sin\theta\,\hat{\mathbf{n}} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \mathbf{i}\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \mathbf{j}\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \mathbf{k}\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

Matrix transformation

Anti-clockwise rotation through an angle, θ, about the origin, $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

[Turn over

Total marks – 100

Attempt ALL questions

1. Write down the binomial expansion of $\left(\frac{2}{y^2}-5y\right)^3$ and simplify your answer. **4**

2. Express $\frac{x^2-6x+20}{(x+1)(x-2)^2}$ in partial fractions. **4**

3. On a suitable domain, a function is defined by $f(x)=\frac{e^{x^2-1}}{x^2-1}$.

Find $f'(x)$, simplifying your answer. **3**

4. The fifth term of an arithmetic sequence is –6 and the twelfth term is –34.

(a) Determine the values of the first term and the common difference. **2**

(b) Obtain algebraically the value of n for which $S_n=-144$. **3**

5. (a) (i) Use Gaussian elimination on the system of equations below to give an expression for z in terms of λ. **4**

$$\begin{aligned} x+2y-z&=-3\\ 4x-2y+3z&=11\\ 3x+y+2\lambda z&=8 \end{aligned}$$

(ii) For what value of λ is this system of equations inconsistent? **1**

(b) Determine the solution of this system when $\lambda=-2{\cdot}5$. **1**

6. Use the substitution $u=5x^2$ to find the exact value of $\int_0^{\frac{1}{\sqrt{10}}}\frac{x}{\sqrt{1-25x^4}}\,dx$. **6**

MARKS

7. Matrices P and Q are defined by $P = \begin{pmatrix} x & 2 \\ -5 & -1 \end{pmatrix}$ and $Q = \begin{pmatrix} 2 & -3 \\ 4 & y \end{pmatrix}$, where $x, y \in \mathbb{R}$.

(a) Given the determinant of P is 2, obtain:

(i) The value of x. **1**

(ii) P^{-1}. **1**

(iii) $P^{-1}Q'$, where Q' is the transpose of Q. **2**

(b) The matrix R is defined by $R = \begin{pmatrix} 5 & -2 \\ z & -6 \end{pmatrix}$, where $z \in \mathbb{R}$.

Determine the value of z such that R is singular. **2**

8. Use the Euclidean algorithm to find integers a and b such that $1595a + 1218b = 29$. **4**

9. Solve $\dfrac{dy}{dx} = e^{2x}\left(1 + y^2\right)$ given that when $x = 0$, $y = 1$.

Express y in terms of x. **5**

10. S_n is defined by $\displaystyle\sum_{r=1}^{n}\left(r^2 + \frac{1}{3}r\right)$.

(a) Find an expression for S_n, fully factorising your answer. **2**

(b) Hence find an expression for $\displaystyle\sum_{r=10}^{2p}\left(r^2 + \frac{1}{3}r\right)$ where $p > 5$. **2**

[Turn over

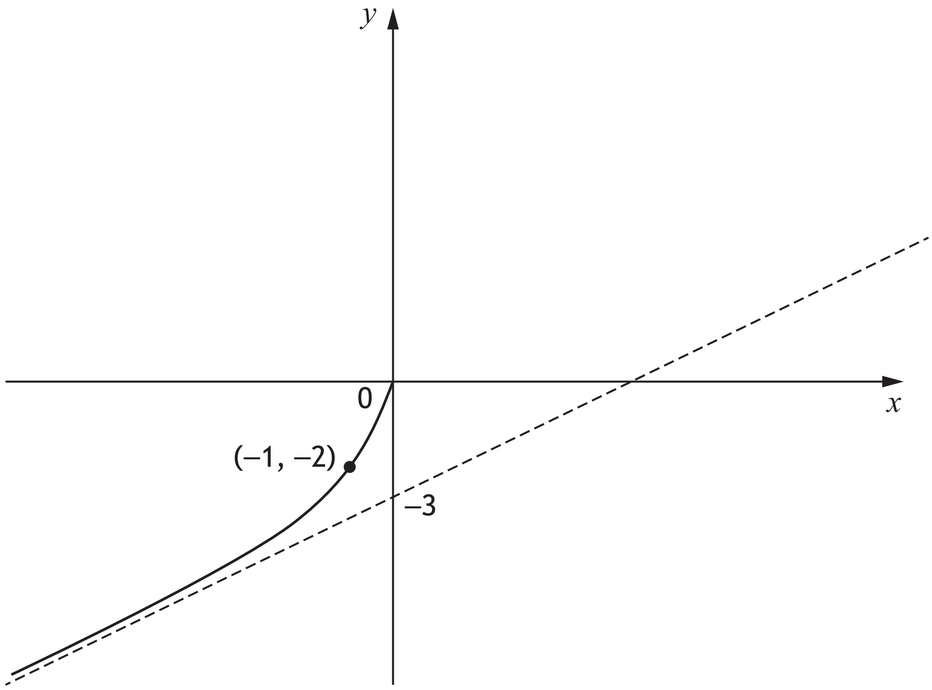
MARKS

11. Given $y = x^{2x^3+1}$, use logarithmic differentiation to find $\frac{dy}{dx}$.

Write your answer in terms of x. **5**

12. In the diagram below part of the graph of $y = f(x)$ has been omitted.

The point (–1, –2) lies on the graph and the line $y = \frac{1}{2}x - 3$ is an asymptote.



Given that $f(x)$ is an odd function:

(a) Copy and complete the diagram, including any asymptotes and any points you know to be on the graph. **2**

(b) $g(x) = |f(x)|$. On a separate diagram, sketch $g(x)$.

Include known asymptotes and points. **2**

(c) State the range of values of $f'(x)$ given that $f'(0) = 2$. **1**

13. Let n be an integer.

Using proof by contrapositive, show that if n^2 is even, then n is even. **4**

14. Find the particular solution of the differential equation

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 8\sin x + 19\cos x$$

given that $y = 7$ and $\frac{dy}{dx} = \frac{1}{2}$ when $x = 0$. **10**

15. (a) A beam of light passes through the points B(7, 8, 1) and T(–3, –22, 6).

Obtain parametric equations of the line representing the beam of light. **2**

(b) A sheet of metal is represented by a plane containing the points P(2, 1, 9), Q(1, 2, 7) and R(–3, 7, 1).

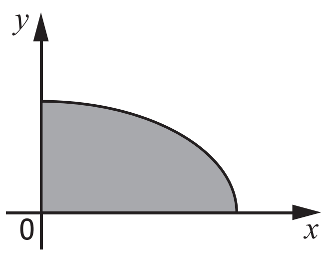
Find the Cartesian equation of the plane. **4**

(c) The beam of light passes through a hole in the metal at point H.

Find the coordinates of H. **3**

16. On a suitable domain, a curve is defined by the equation $4x^2 + 9y^2 = 36$.

A section of the curve in the first quadrant, illustrated in the diagram below, is rotated 360° about the ***y*-axis**.



Calculate the exact value of the volume generated. **5**

[Turn over for next question

MARKS

17. The complex number $z=2+i$ is a root of the polynomial equation $z^4-6z^3+16z^2-22z+q=0$, where $q\in\mathbb{Z}$.

(a) State a second root of the equation. **1**

(b) Find the value of q and the remaining roots. **6**

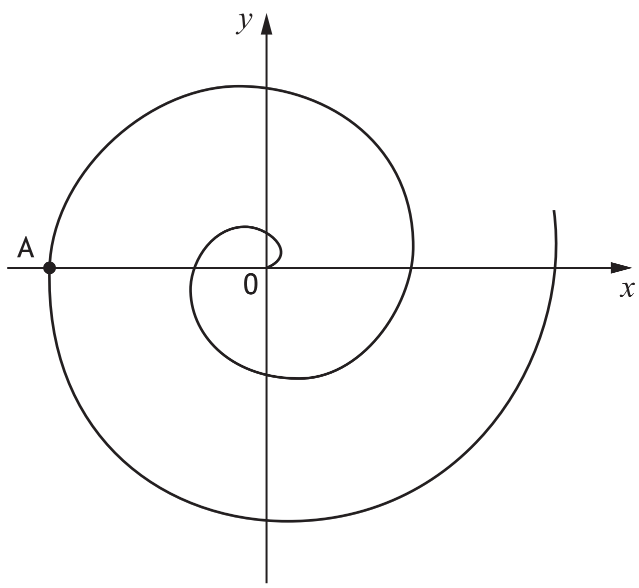
(c) Show the solutions to $z^4-6z^3+16z^2-22z+q=0$ on an Argand diagram. **1**

18. The position of a particle at time t is given by the parametric equations

$x=t\cos t,\quad y=t\sin t,\quad t\geq 0.$

(a) Find an expression for the instantaneous speed of the particle. **5**

The diagram below shows the path that the particle takes.



(b) Calculate the instantaneous speed of the particle at point A. **2**

[END OF QUESTION PAPER]